Computations with large numbers

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Abstract. Since any personal computer has a limited range of integer values, therefore, it will result in an integer overflow when a program tries to compute a value larger than machine’s maximum value. We will discuss a workable algorithm that will be able to deal with any large numbers without getting an integer overflow.

Keywords: Integer overflow.

§1. Introduction
Any programmer knows that when a variable is declared to be of integer type, it has a maximum value. It will result in an integer overflow when a program is to compute an integer value larger than your machine’s maximum value. Some machine gives you an error message when overflow occurs, but others don’t. Therefore, it is necessary to prevent it from occurring. It would be nice to find an algorithm that will be able to deal with any large number without getting an integer overflow on any machine. In the following, we will provide an algorithm that will fulfill such goal.

§2. Algorithm
Our approach is simple and easy for implementation. The idea is to take any integer input as a string, which can be as long as you wish. For instance, if you have to calculate

$$12345678912345678998162727272822221234 + 1234598726262626262626262626262681818819982$$

you can view both numbers as string, then manipulate them either digit by digit or block by block. The following algorithm Add (A, B) is an algorithm of digit by digit.

Algorithm Add (A, B):
A = A_n A_{n-1} A_{n-2} \cdots A_1 A_0,
where A_i is the unit digit, A_i is the tenth digit, and so on
B = B_n B_{n-1} B_{n-2} \cdots B_1 B_0,
where B_i is the unit digit, B_i is the tenth digit, and so on
(if not of the same length, fill it out by zeros to the left)
Initialize Carry = 0;
For (i = 0; i < n; i++)
if A_i + B_i + Carry > 9,
then \{C_i = A_i + B_i + Carry - 10, Carry = 1\}
else \{C_i = A_i + B_i + Carry, Carry = 0\}
if Carry > 0, then C_{n+1} = Carry else C_{n+1} = "";
Output: C = C_{n+1} C_n C_{n-1} \cdots C_1 C_0
Considering the efficiency for large numbers, we might divide each number into blocks of length \( k \) that can be determined by the machine’s maximum integer, and then use regular operation of addition to carry out the sum of each block, and then concatenate the sums of blocks to get the sum. The following algorithm demonstrates the idea of block by block.

Algorithm Add2 (A, B)

Assume that \( k \) is a positive whole number;

\[
A = A_p \times 10^k + A_{p-1} \times 10^{(p-1)k} + A_{p-2} \times 10^{(p-2)k} + \cdots + A_1 \times 10^k + A_0;
\]

\[
B = B_q \times 10^k + B_{q-1} \times 10^{(q-1)k} + B_{q-2} \times 10^{(q-2)k} + \cdots + B_1 \times 10^k + B_0;
\]

where \( A_j (0 \leq j \leq p) \) and \( B_j (0 \leq j \leq q) \) are whole numbers of \( k \) or fewer digits;

Each sum \( A_j + B_j + 1 (j = 0, 1, 2, \ldots, \min(p,q)) \) is less than the machine’s max;

1. Use regular addition to find each sum \( S_j = A_j + B_j (j = 0, 1, \ldots, \min(p,q)) \);
   - If \( S_j \_\text{length} = k + 1, \text{then Carry} = 1, \text{otherwise Carry} = 0; \)
   - Set \( S_j = A_j (\text{if } p > q) \) or \( B_j (\text{if } p \leq q) \) for \( j = \min(p,q) + 1 \) to \( \max(p,q) \);

2. If \( \text{Carry} = 1 \) then \( S_0 = S_0 - 10^k; \) pach \( k - S_0 \_\text{length} \) 0s to the left of \( S_0; \)
   - \( C = "\& S_0; \)

3. For \( j = 1, 2, \ldots, \max(p,q) \)
   - \( S_j = S_j + \text{Carry}; \)
     - if \( S_j \_\text{length} < k + 1 \) then
       - if \( j < \max(p,q) \) then pach \( k - S_j \_\text{length} \) 0s to the left of \( S_j; \)
         - \( C = S_j \& C; \) Carry = 0;
       - else
         - \( \text{Carry} = 1; S_j = S_j - 10^k; \)
           - pach \( k - S_j \_\text{length} \) 0s to the left of \( S_j; \)
         - \( C = S_j \& C; \)
   - }

4. If \( \text{Carry} = 1 \) then \( C = "1" \& C; \)

5. Output \( C \) as the sum \( A + B. \)
Likewise, we can do subtraction in a similar way. The following is an algorithm for subtraction in digit by digit.

Algorithm Subtract (A, B):

\[ A = A_{n}A_{n-1}A_{n-2} \cdots A_{1}A_{0}, \]
where \( A_{0} \) is the unit digit, \( A_{1} \) is the tenth digit, and so on

\[ B = B_{n}B_{n-1}B_{n-2} \cdots B_{1}B_{0}, \]
where \( B_{0} \) is the unit digit, \( B_{1} \) is the tenth digit, and so on

(if not of the same length, fill it out by zeros to the left)

If \( A \geq B \) then sign ="+"
else sign ="-"

Initialize Borrow = 0;

For \( i = 0; i < n; i += 1 \)

if \( A_{i} - Borrow < B_{i} \),
then \( \{ C_{i} = A_{i} - Borrow + 10 - B_{i}, Borrow = 1 \} \)
else
\( \{ C_{i} = A_{i} - Borrow - B_{i}, Borrow = 0 \} \)

Output: if (sign =="-") then \( -C_{n}C_{n-1} \cdots C_{0} \)
else \( C_{n}C_{n-1} \cdots C_{0} \)

We can also do subtraction block by block as shown in Algorithm Subtract2 (A, B).

Algorithm Subtract2 (A, B):

Assume that \( k \) is a positive whole number;

\[ A = A_{p} \times 10^{p} + A_{p-1} \times 10^{(p-1)} + \cdots + A_{1} \times 10^{k} + A_{0}; \]
\[ B = B_{q} \times 10^{q} + B_{q-1} \times 10^{(q-1)} + \cdots + B_{1} \times 10^{k} + B_{0}; \]

where \( A_{j} (j = 0, 1, 2, 3, \ldots, p - 1) \) and \( B_{j} (j = 0, 1, 2, 3, \ldots, q - 1) \) are whole numbers of \( k \) digits (pach 0s if necessary);
\( A_{p} \) and \( B_{q} \) are whole numbers of \( k \) or fewer digits;
\( A_{j} \) and \( B_{j} \) are less than the machine’s max;

(1) if \( A = B \) then
output \( D = 0; \) Exit;
else if \( A > B \) then sign ='+';
else \{ sign ='-'; swap A and B;\}
(2) Assume $A > B$;
   Initialize $\text{Borrow} = 0$;
   Use regular subtraction to find each difference:
   For $(j = 0 \text{ to } \max(p, q))$
   if $(j \leq \min(p, q))$ then
      if $(A_j - \text{Borrow} > B_j)$ then $(D_j = A_j - B_j - \text{Borrow}; \text{Borrow} = 0;)$
      else $(D_j = A_j + 10^k - B_j - \text{Borrow}; \text{Borrow} = 1;)$
   else
      if $(A_j - \text{Borrow} > 0)$ then $(D_j = A_j - \text{Borrow}; \text{Borrow} = 0;)$
      else $(D_j = A_j + 10^k - \text{Borrow}; \text{Borrow} = 1;)$
   (3) Let $m = \max(p, q)$;
   (4) For $(j = 0 \text{ to } m - 1)$, patch $(k - D_j$, length) 0s to the left of $D_j$;
   (5) $D = "\"$;
      for $(j = 0 \text{ to } m)$ $D = D_j & D$;
   (6) if $(\text{sign} = \text{\textminus})$ then $D = \text{sign} & D$;
   (7) Output $D$ as the difference $A - B$.

For multiplication, we can do either digit wise or block wise. The following is an algorithm for multiplication in block wise.

Algorithm Multiply $(A, B)$:
   Assume that
   $A = A_p \times 10^{ek} + A_{p-1} \times 10^{(p-1)k} + A_{p-2} \times 10^{(p-2)k} + \cdots + A_i \times 10^k + A_0$;
   $B = B_q \times 10^{ek} + B_{q-1} \times 10^{(q-1)k} + B_{q-2} \times 10^{(q-2)k} + \cdots + B_j \times 10^k + B_0$;
   where $A_i, B_j (0 \leq i \leq p; 0 \leq j \leq q)$ are whole numbers of $k$ or fewer digits.
   Each product $A_iB_j$ is less than the machine's max;
   (1) Use regular multiplication to find each product $A_iB_j$;
   (2) Attaching $(i + j)k$ zeros to the end of the product $A_iB_j$;
   (3) Use the Add $(A, B)$ or Add 2 $(A, B)$ to sum up all products;
   (4) Output the sum as the product of $A$ and $B$.

In order to check divisibility and primality of integers, we need an algorithm of division. We can proceed as follow:

Algorithm Divide $(A, B)$:
   $A = A_nA_{n-1}A_{n-2} \cdots A_iA_0$,
   where $A_i$ is the unit digit, $A_i$ is the tenth digit, and so on
   $B = B_mB_{m-1}B_{m-2} \cdots B_iB_0$,
   where $B_0$ is the unit digit, $B_i$ is the tenth digit, and so on
(1) If $A < B$, then Output quotient = 0, and remainder = $A$.
(2) If $A = B$, then Output quotient = 1, and remainder = 0.
(3) If $A > B$, then
- Dividend = $A$, divisor = $B$,
- $K =$ the length of $B$;

While (Dividend > divisor)
{
    $D1 =$ the first $K$ digits of Dividend from the left.
    $D2 =$ the substring of Dividend starting from the $(K+1)$ to the end
    If ($D1 >=$ divisor) then
        \begin{itemize}
        \item Find the greatest non-negative integer $k$ such that $k \times (\text{divisor}) \leq D1$;
        \item $q_{count} = k$;
        \item Update Dividend = Subtract ($D1$, Multiply ($k$, divisor)) & $D2$;
        \end{itemize}
    Else
    \begin{itemize}
    \item $q_{count} = 0$;
    \item $D1 =$ the first $K + 1$ digits of Dividend
    \item $D2 =$ the substring of Dividend starting from the $(K+2)$ to the end
    \end{itemize}
    \begin{itemize}
    \item Find the greatest non-negative integer $k$ such that $k \times (\text{divisor}) \leq D1$;
    \item $q_{count} = k$;
    \item Update Dividend = Subtract ($D1$, Multiply ($k$, divisor)) & $D2$;
    \end{itemize}
}\}
If (Dividend = divisor) then
\begin{itemize}
\item $q_{count} = 1$; \quad \text{Remainder} = 0;
\end{itemize}
Else
\begin{itemize}
\item \text{Remainder} = Dividend;
\end{itemize}

Output: quotient = $q_0 q_1 q_2 \cdots q_{count-1}$

\text{remainder} = \text{Remainder}

Once we implement Divide $(A, B)$, we can define modulo function $\text{Mod} (A, B)$, which returns the remainder when $A$ is divided by $B$. Therefore, we can check divisibility and primality of integers.

§3. Applications

It is very common that whenever we teach Number Theory, we need display some large numbers such as Fibonacci primes and Mersenne numbers by using computer. If we are not careful, we will run into an integer overflow. We can use the algorithms developed in section 2 above to eliminate such errors. As a demonstration, we give two examples in the following. The first example is a comparison of displays of large numbers on computers with or without using our algorithms.
Example 1. It is well known\textsuperscript{22} that the 3\textsuperscript{rd}, 4\textsuperscript{th}, 7\textsuperscript{th}, 11\textsuperscript{th}, 13\textsuperscript{th}, 17\textsuperscript{th}, 23\textsuperscript{rd}, 29\textsuperscript{th}, 43\textsuperscript{rd}, 47\textsuperscript{th}, 83\textsuperscript{rd}, 131\textsuperscript{st}, 137\textsuperscript{th}, 359\textsuperscript{th}, 431\textsuperscript{st}, 433\textsuperscript{rd}, 449\textsuperscript{th}, 509\textsuperscript{th}, 569\textsuperscript{th}, 571\textsuperscript{st}, 2971\textsuperscript{st}, 4723\textsuperscript{rd}, 5387\textsuperscript{th}, 9311\textsuperscript{th}, 9677\textsuperscript{th}, 14431\textsuperscript{st}, 25561\textsuperscript{st}, 30757\textsuperscript{th}, 30757\textsuperscript{th}, 35999\textsuperscript{th}, and 81839\textsuperscript{th} terms of Fibonacci sequence are primes. It is easy to display any of the first few small ones using a computer. However, it is impossible to display a large one without getting an integer overflow. In the table 1, it was shown that the display via JavaScript without using our algorithm will either loss precisions or get a scientific notation that unfortunately results in a real number instead of an integer. A live demonstration is available on our website at http://cims.clayton.edu/whong/tools/Demo4BigNumber.htm.

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<th>The Display using our algorithm</th>
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</table>

Table 1 Fibonacci Primes.
Example 2. It is well known\(^1\) that the 257\(^{th}\) Mersenne number is a prime. To display the 257\(^{th}\) Mersenne number, we can make a simple loop as follows:

```
Count = 0;
X = 1;
While (Count Not Equal to 257)
{ X = Multiply(X,"2");
Z = Subtract(X,"1");
Count++;
}
Output Z;
```

The 257\(^{th}\) Mersenne number is

\[231584178474632390847141970017375815706539969331281128078915168015826259279871.\]

§4. Conclusion

It is very common that an error of integer overflow occurs in scientific computation. To eliminate such an error, it is a good idea to implement algorithms as we did in this note in either digit by digit or block by block so that programs will be able to handle situations intelligently for larger integers. The algorithm given in this note can be easily applied to many scientific computations such as RSA\(^3\) encryption and searching for large primes.

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References

