Applications of Persistent Homology to Ricci Flow on S^2 and S^3

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Joint Mathematics Meetings
7 Jan 2016

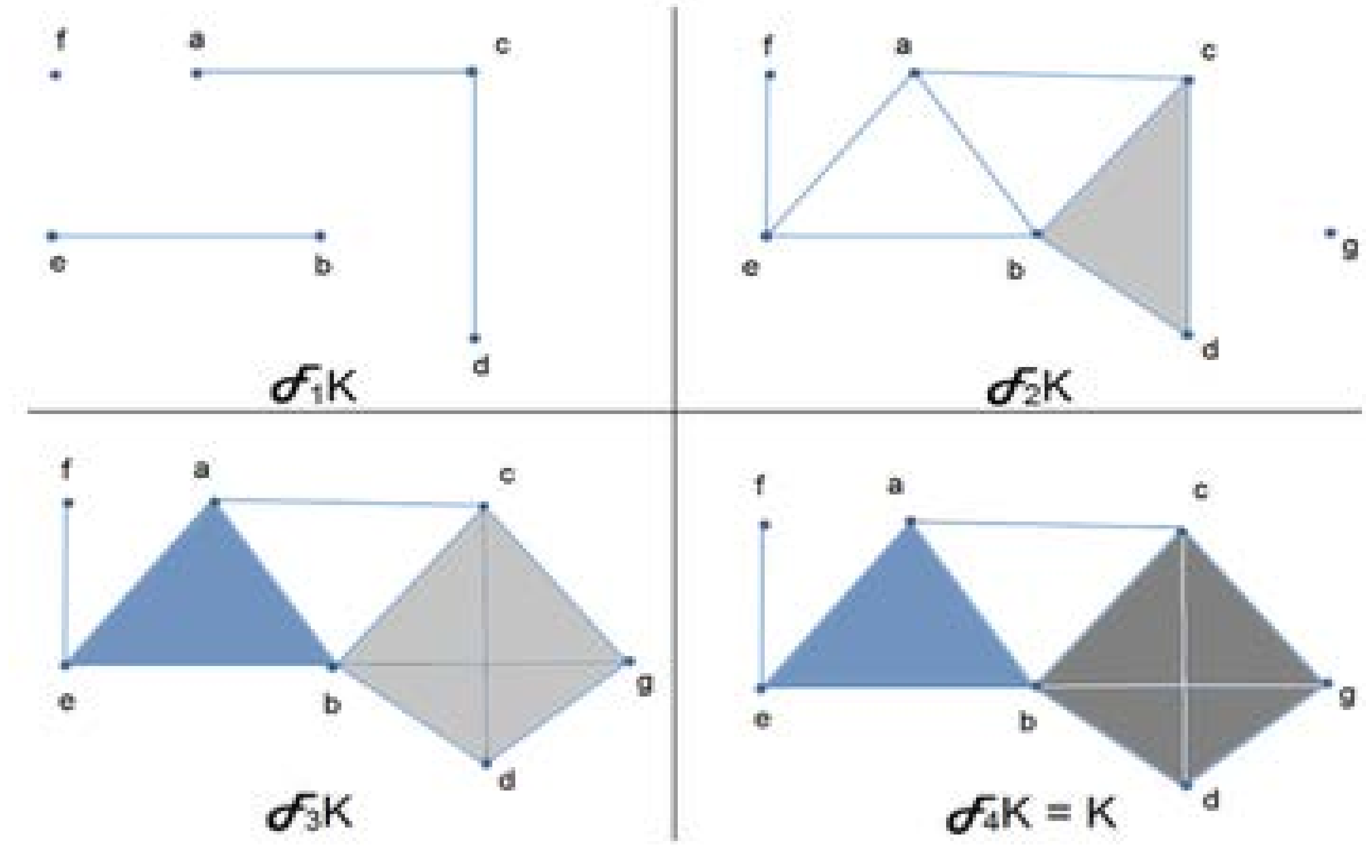
Objective: Obtain persistent homological characterization of singularity formation in Ricci flow

Approach

- Persistent Homology: Builds increasing family of simplicial complexes K

$$\varnothing = F_0 K \xrightarrow{i_{0,1}} F_1 K \xrightarrow{i_{1,2}} \dots \xrightarrow{i_{N-2,N-1}} F_{N-1} K \xrightarrow{i_{N-1,N}} F_N K = K$$

- Tracks persistent topological features



Objective: Obtain persistent homological characterization of singularity formation in Ricci flow

Approach

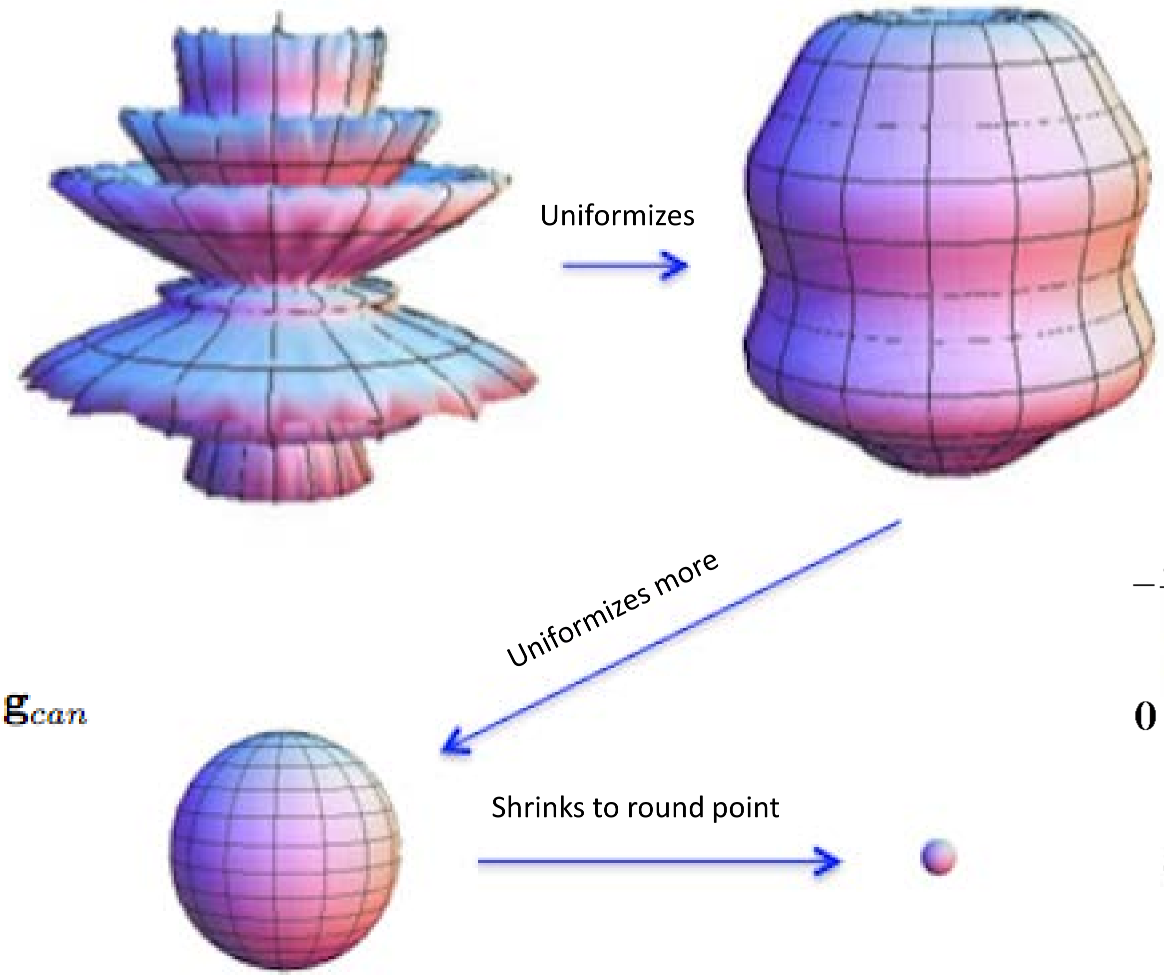
- Ricci Flow: Diffuses ``irregularities'' in the metric

$$\frac{\partial g_{ij}}{\partial t} = -2R_{ij}.$$

- Poincaré conjecture
- Discrete formulations useful for imaging

Ricci Flow: Rotational Solid (2D)
(Uniformization)

Dimpled Sphere



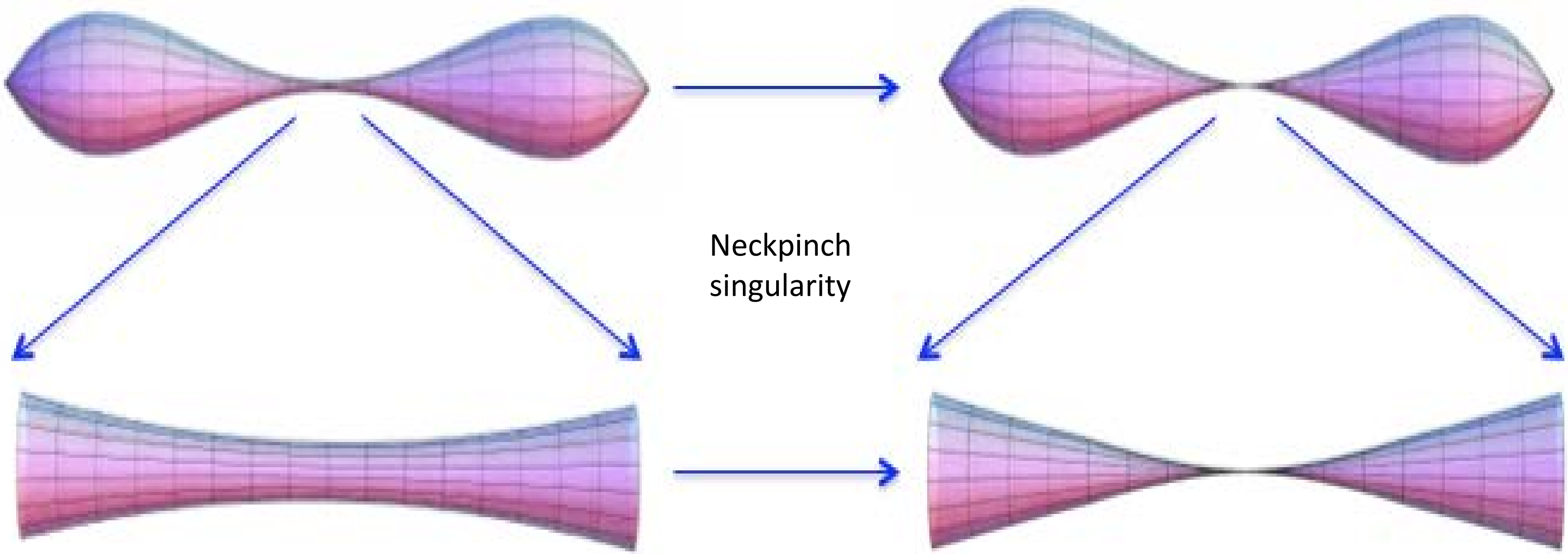
$$S^2$$

$$\mathbf{g}(t) = r^2(\theta, t)\mathbf{g}_{can}$$

$$-\frac{\pi}{2} < \theta < \frac{\pi}{2}$$

$$0 \leq \phi < 2\pi$$

Ricci Flow: Symmetric Dumbbell (3D)
(Neckpinch)



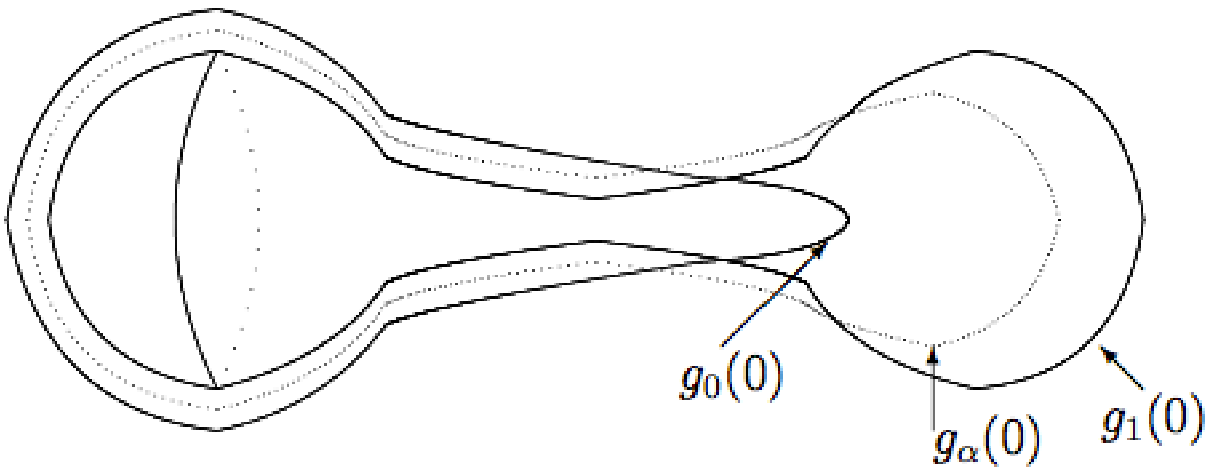
$$S^3 \setminus \{N, S\} \cong S^2 \times (-c, c)$$

$$\mathbf{g}(t) = \varphi^2(x,t)dx^2 + \psi^2(x,t)\mathbf{g}_{can} \qquad \mathbf{g}(t) = ds^2 + \psi^2(s,t)\mathbf{g}_{can}$$

$$-c < x < c \qquad 0 \leq \theta < 2\pi$$

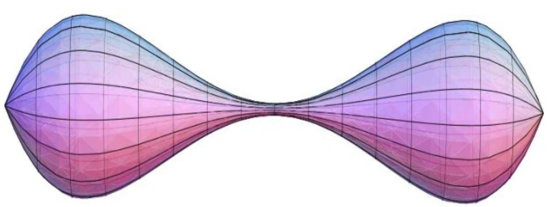
$$s(x,t) = \int_0^x \varphi(x', t)dx'$$

Ricci Flow: Interpolative Dumbbells

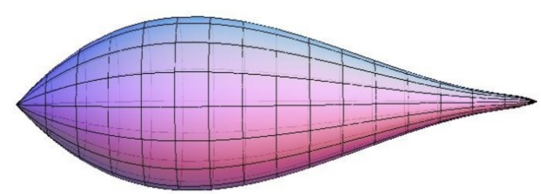


R. Hamilton (1995);
H-L Gu, X-P Zhu (2007)

$$\psi(x) = \frac{-\mu a\left(\frac{x}{\mu}+\frac{\pi}{2}\right)\left(\left(\frac{x}{\mu}-(1-\alpha)L\right)^2+h(\alpha,k)\right)\left(\frac{x}{\mu}-\left(\left(\frac{\pi}{2}-L\right)\alpha+L\right)\right)}{\left(b-\frac{x}{\mu}\right)^{(1-\alpha)}}$$



$$\alpha \in [0,1]$$

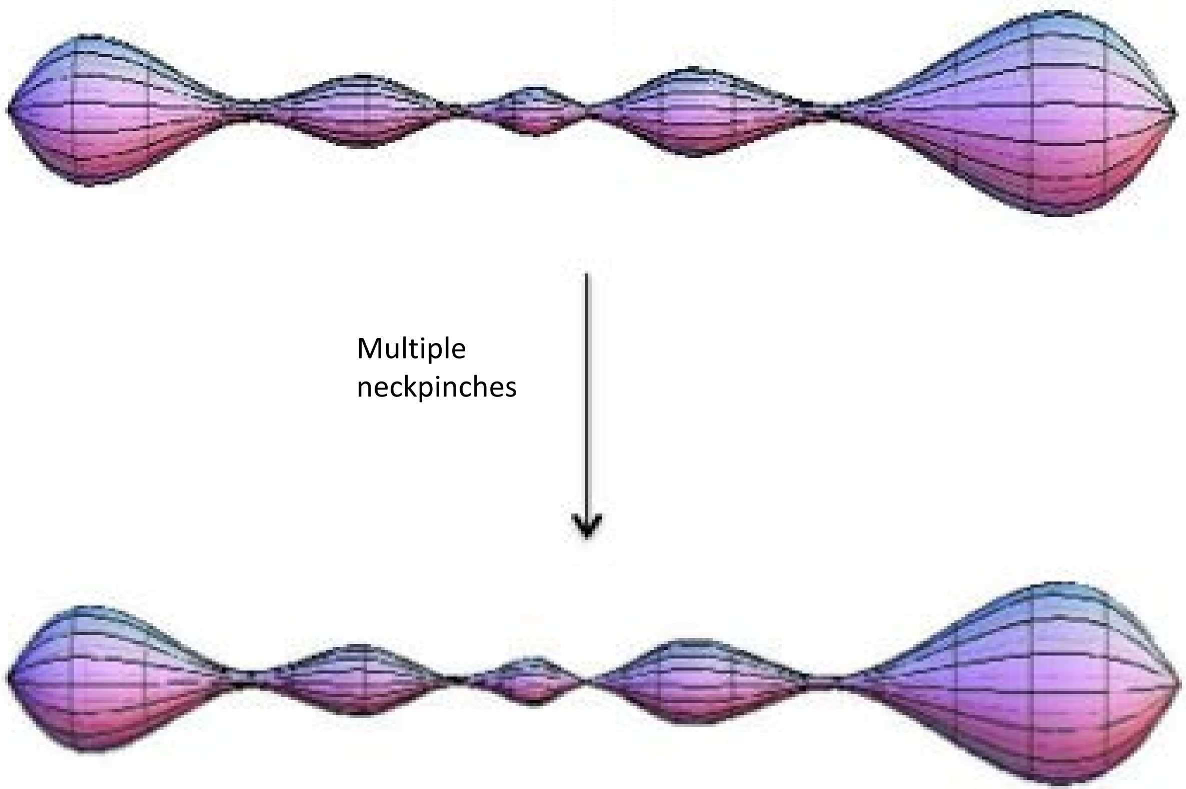


Symmetric dumbbell; satisfies (1) – (5); Type-I (rapidly forming) singularity

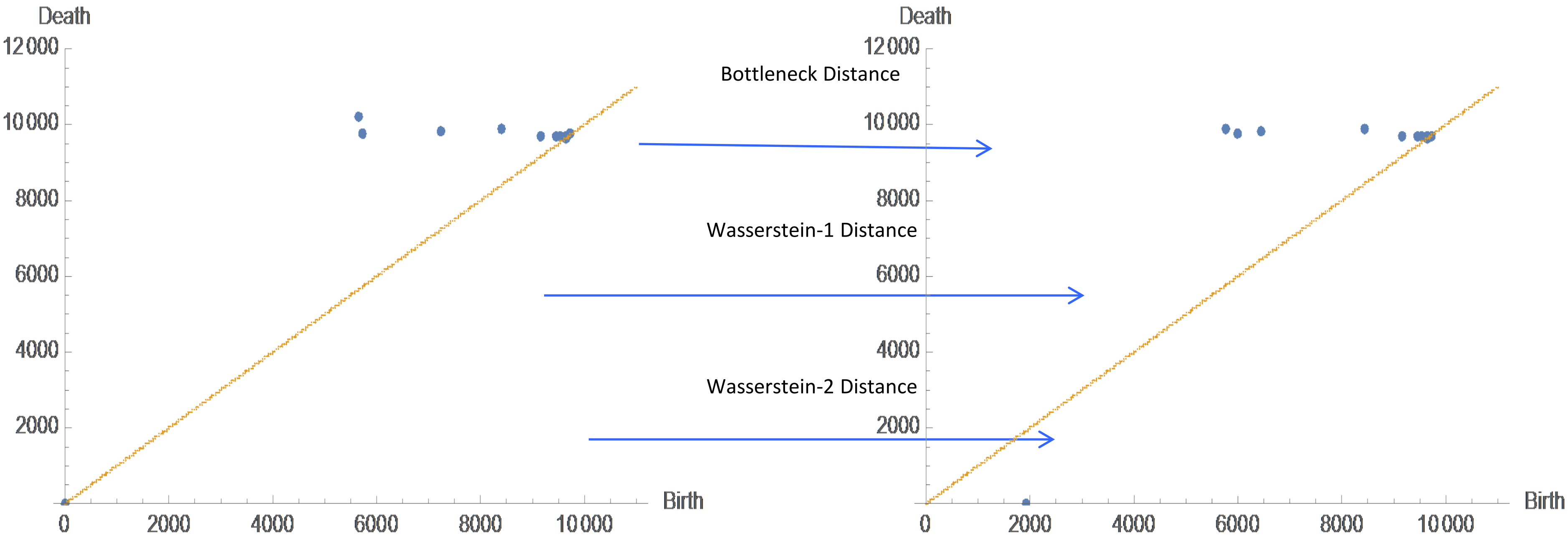
Degenerate dumbbell; satisfies (1) – (3); Type-II (slowly forming) singularity

Ricci Flow: Dimpled Dumbbell (3D)
(Neckpinch)

Genetically Modified Peanut (Angenent & Knopf)



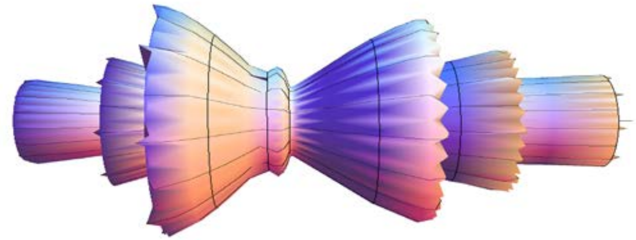
Persistent Homology: Distances Between Persistence Diagrams



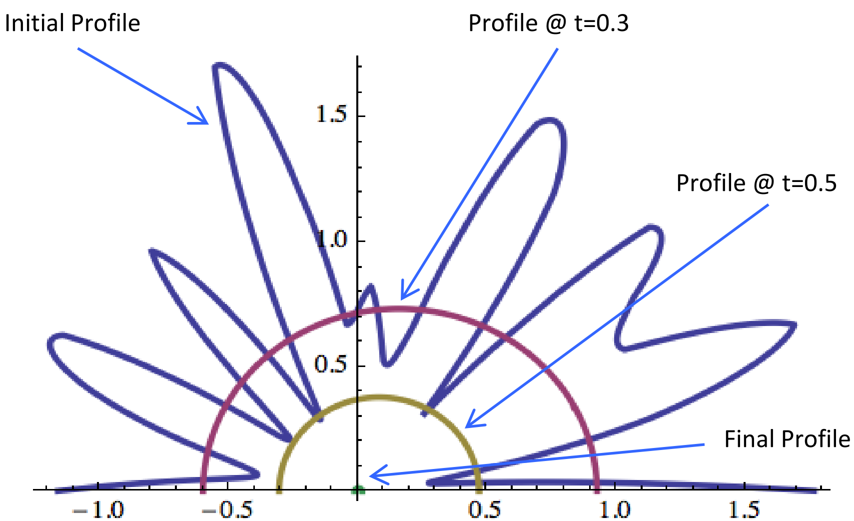
- Bottleneck distance: measures largest single difference between persistence diagrams

- Wasserstein distances: measure all differences between diagrams with higher number indicating less sensitivity to small changes (noise)

Persistent Homology: Dimpled Sphere

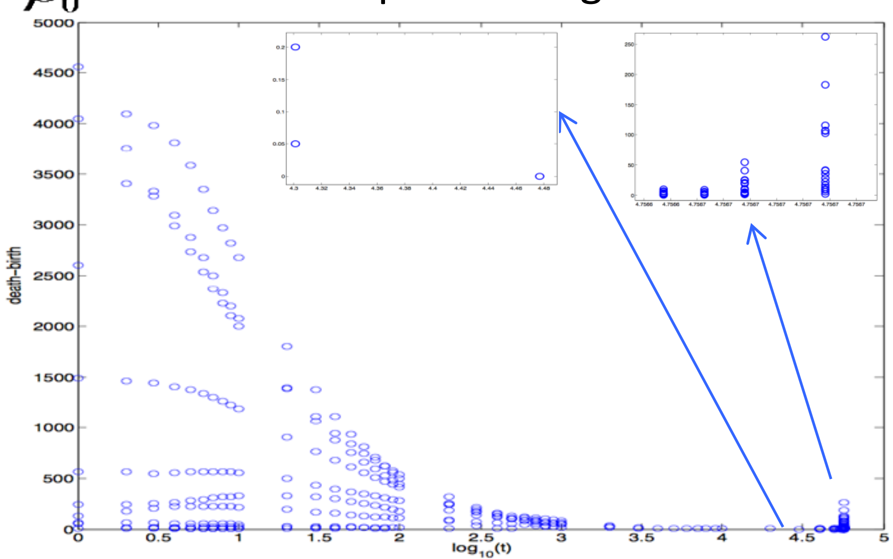


Radial Profiles

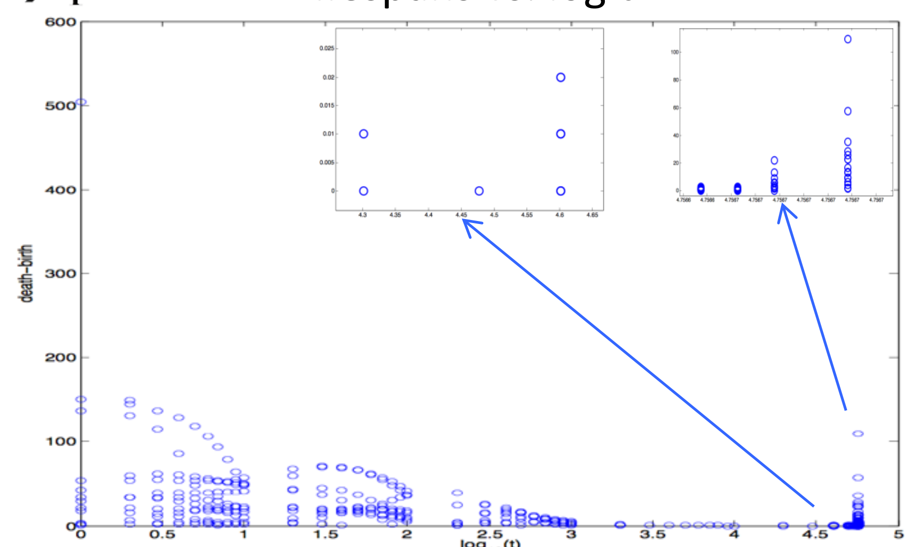


Rotational Solid	β_0	β_1
$(\mathrm{PD}^i, \mathrm{PD}^{i+1})$	$(d_B,\ d_{W1},\ d_{W2})$	$(d_B,\ d_{W1},\ d_{W2})$
$i = 18$	$(5.42 \times 10^1,\ 1.81 \times 10^2,\ 8.09 \times 10^1)$	$(1.04 \times 10^1,\ 4.35 \times 10^1,\ 1.93 \times 10^1)$
$i = 21$	$(4 \times 10^1,\ 1.69 \times 10^2,\ 7.06 \times 10^1)$	$(3.08 \times 10^1,\ 7.89 \times 10^1,\ 3.76 \times 10^1)$
$i = 37$	$(1.31,\ 2.43,\ 1.72)$	$(0.18,\ 0.19,\ 0.1803)$

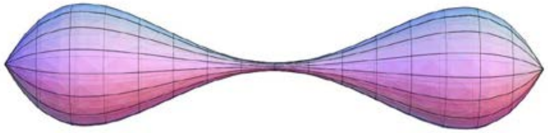
β_0 Lifespans vs. log t



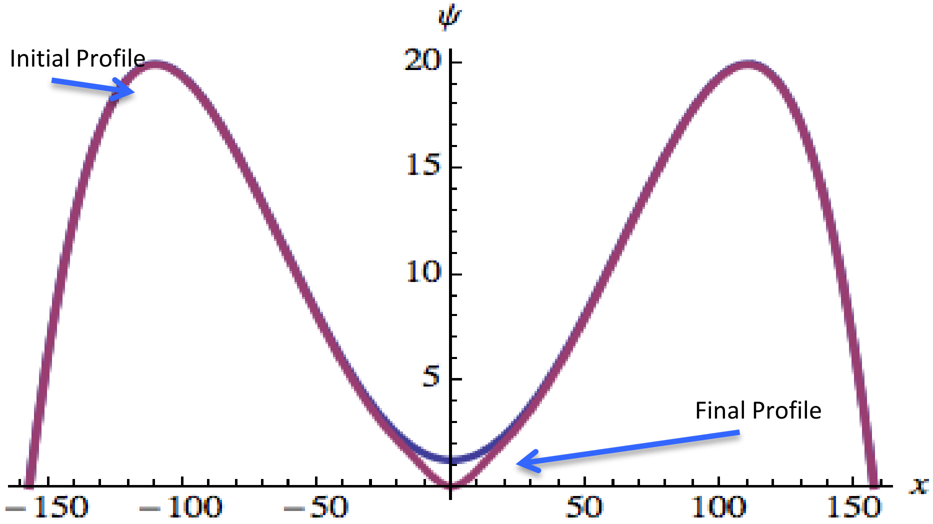
β_1 Lifespans vs. log t



Persistent Homology: Symmetric Dumbbell



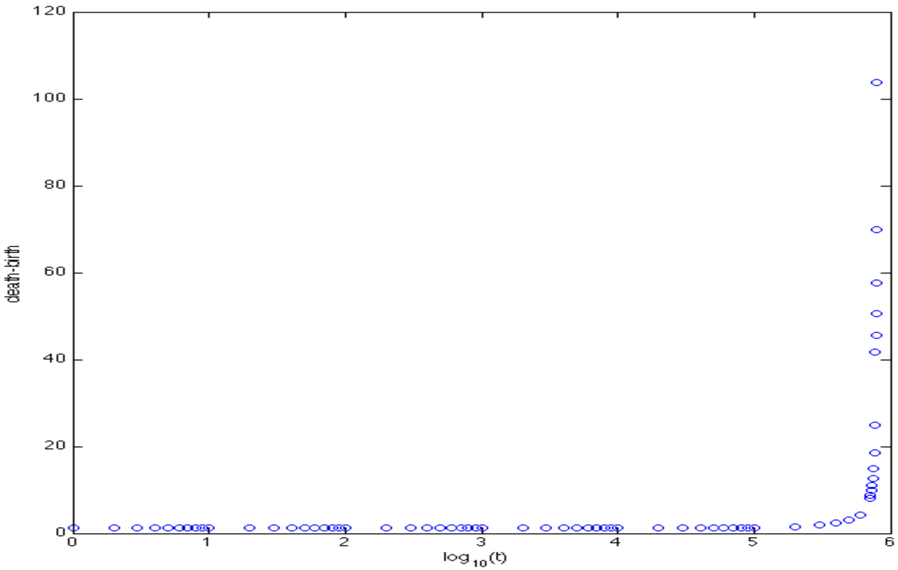
Radial Profiles



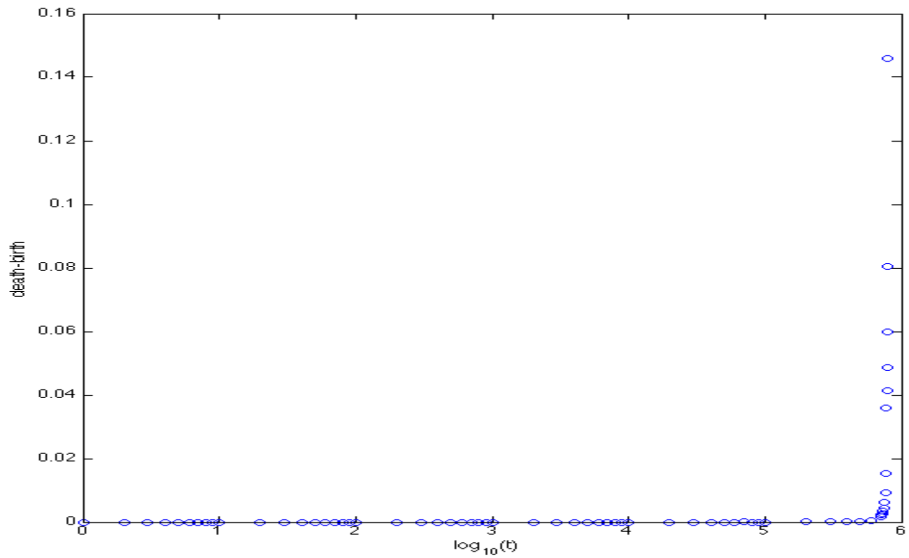
50×50	β_0	β_1
$(\mathrm{PD}^i, \mathrm{PD}^{i+1})$	$(\mathrm{d}_B,\ \mathrm{d}_{W1},\ \mathrm{d}_{W2})$	$(\mathrm{d}_B,\ \mathrm{d}_{W1},\ \mathrm{d}_{W2})$
$i = 1$	0	0
$i = 17$	2×10^{-5}	2×10^{-5}
$i = 33$	1.46×10^{-3}	1.46×10^{-3}
$i = 49$	6.95×10^{-1}	6.95×10^{-1}
$i = \max$	3.37×10^{1}	3.37×10^{1}

100×100	β_0	β_1
$(\mathrm{PD}^i, \mathrm{PD}^{i+1})$	$(\mathrm{d}_B,\ \mathrm{d}_{W1},\ \mathrm{d}_{W2})$	$(\mathrm{d}_B,\ \mathrm{d}_{W1},\ \mathrm{d}_{W2})$
$i = 1$	0	0
$i = 17$	2×10^{-5}	2×10^{-5}
$i = 33$	1.58×10^{-3}	1.58×10^{-3}
$i = 49$	8.22×10^{-1}	8.22×10^{-1}
$i = \max$	4.61×10^{2}	4.62×10^{2}

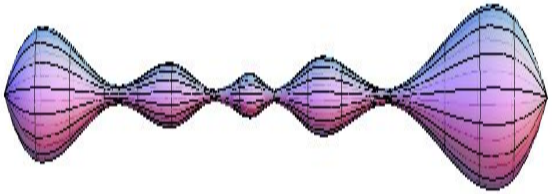
β_0 Lifespans vs. log t



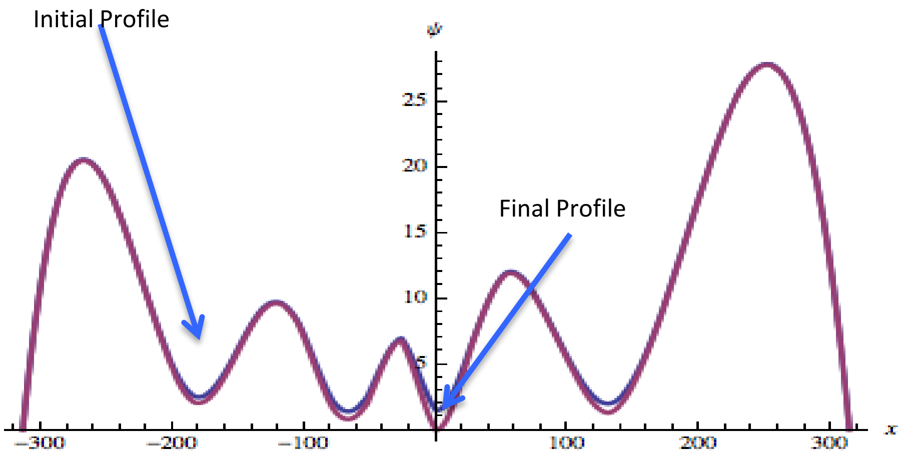
β_1 Lifespans vs. log t



Persistent Homology: Dimpled Dumbbell

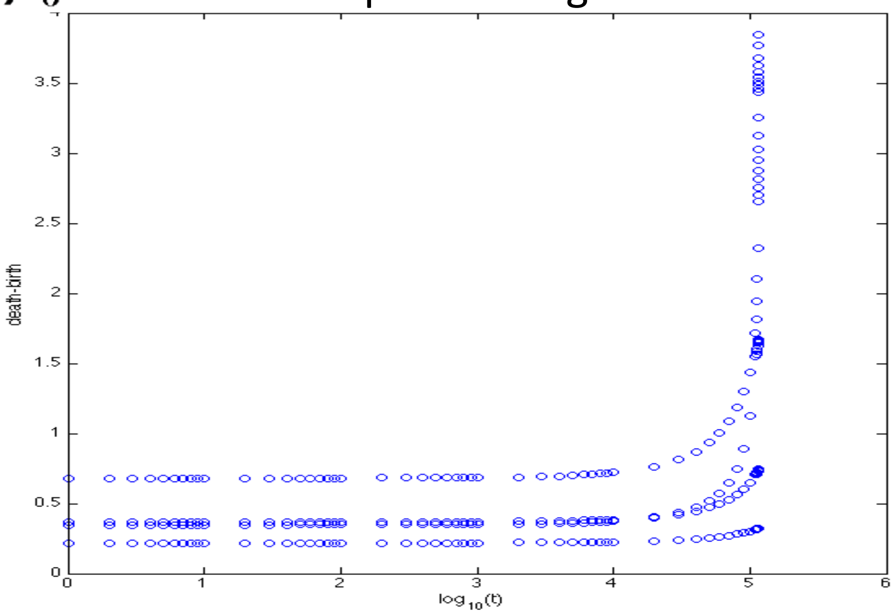


Radial Profiles

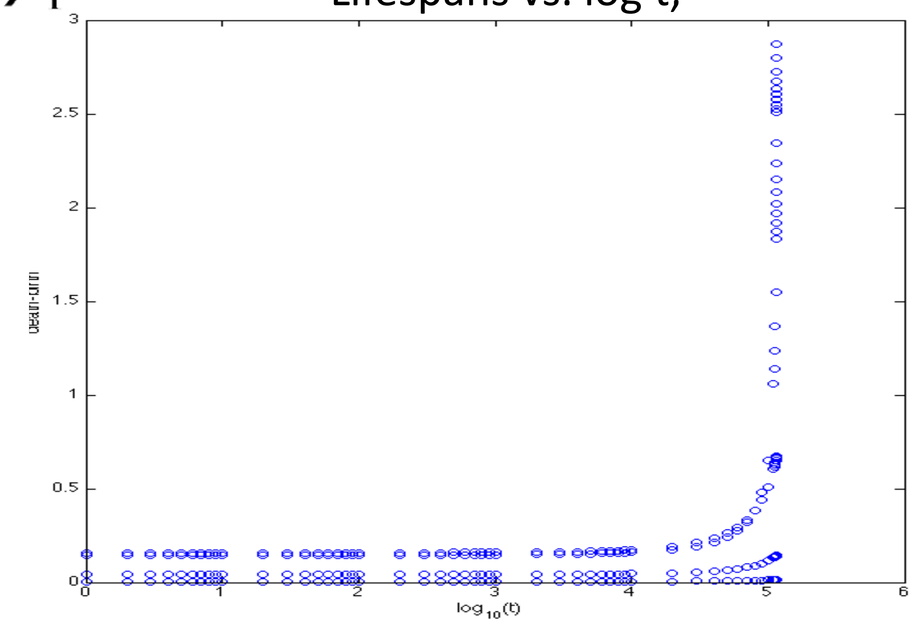


GMP 50 x 10	β_0	β_1
(PD^i, PD^{i+1})	$(d_B,\ d_{W1},\ d_{W2})$	$(d_B,\ d_{W1},\ d_{W2})$
$i = 1$	$(3 \times 10^{-6},\ 8 \times 10^{-6},\ 4.24 \times 10^{-6})$	$(5 \times 10^{-6},\ 1.2 \times 10^{-5},\ 6.78 \times 10^{-6})$
$i = 18$	$(3.8 \times 10^{-5},\ 8.3 \times 10^{-5},\ 4.76 \times 10^{-5})$	$(5.2 \times 10^{-5},\ 1.13 \times 10^{-4},\ 6.59 \times 10^{-5})$
$i = 35$	$(4.12 \times 10^{-3},\ 9.03 \times 10^{-3},\ 5.18 \times 10^{-3})$	$(5.7 \times 10^{-3},\ 1.22 \times 10^{-2},\ 7.25 \times 10^{-3})$
$i = 52$	$(4.72 \times 10^{-2},\ 5.02 \times 10^{-2},\ 4.73 \times 10^{-2})$	$(8.74 \times 10^{-2},\ 6.61 \times 10^{-1},\ 8.75 \times 10^{-2})$
$i = \max$	$(8.097 \times 10^{-2},\ 8.14 \times 10^{-2},\ 8.098 \times 10^{-2})$	$(1.53 \times 10^{-1},\ 1.54 \times 10^{-1},\ 1.53 \times 10^{-1})$

β_0 Lifespans vs. log t



β_1 Lifespans vs. log t,



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